Comparative Study of Different Methods in Fuzzy Linear Programming Problem

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of

the degree

Master of Science
in
Mathematics

by

Raju Kumar
(MS23MTH021)

under the guidance of

Dr. Vishnu Pratap Singh

Department of Mathematics

Visvesvaraya National Institute of Technology

Nagpur 440010 (India)

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Dedicated To

My Parents and Teachers.

Department of Mathematics
Visvesvaraya National Institute of Technology, Nagpur

Declaration

I, Raju Kumar, hereby declare that the dissertation titled **"Comparative Study of Different Methods in Fuzzy Linear Programming Problem"** is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution/University.

Raju Kumar

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Certificate

This is to certify that the dissertation titled **"Comparative Study of Different Methods in Fuzzy Linear Programming Problem"** is submitted by **Raju Kumar** in partial fulfillment of the requirement for the award of the degree of **Master of Science in Mathematics**, VNIT Nagpur. The work is comprehensive, complete and fit for final evaluation.

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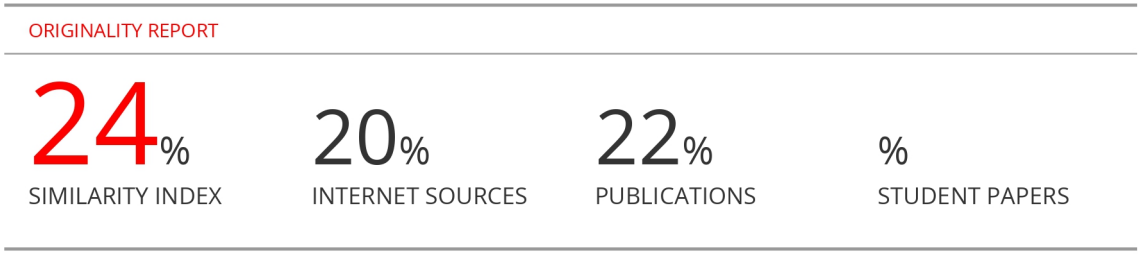

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ABSTRACT

In this study we explore basic concepts of fuzzy set theory and its applications in decision making in fuzzy environment. We learn about fuzzy linear programming. FLP can be regarded as a unique type of decision model.. Then we discuss different types of FLP, including symmetric fuzzy LP (objective function and constraints are fuzzy). For solving such a type of problem, we use the max-min approach. Fuzzy LP problems with a crisp objective function are another kind of problem. (When all or some of the constraints are fuzzy, the goal function is crisp.) that is not symmetric. For solving these types of problem, two approaches are conceivable, the first one is the determination of a fuzzy set ”decision” and the second one is the determination of a crisp set ”maximizing decision”. In this study, we focus on the first solution method (for FLPP with a crisp objective function). The approach helps decision-makers to understand how changing in resources affects the optimal solution. By comparative analysis of methods we understand that for different type of FLPP we can choose different type of solution method.

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1 Introduction

Decision-making lies at the heart of both individual cognition and organizational strategy, forming the backbone of disciplines as varied as economics, management science, artificial intelligence, and public policy. Seminal contributions by Herbert Simon and Victor Vroom introduced foundational concepts such as bounded rationality and normative leadership decision models, respectively, framing how choices are made under constraints of information and cognitive capacity [11, 12]. These efforts coalesced into the broader field of decision theory, which integrates insights from economics, psychology, and operations research to analyze choices under uncertainty, risk, and conflicting objectives [5, 6]. Within this framework, normative decision theory prescribes the ideal of utility maximization and Bayesian updating as objective standards for rational choice, while descriptive decision theory uncovers how real-world decision-makers deviate from these ideals through biases and heuristics [7, 8]. The dual lenses of normative and descriptive approaches have proved indispensable for strategic planning, resource allocation, and public policy formulation, underscoring the interdisciplinary reach of decision theory [17, 18].

Linear programming (LP), a particular decision model intended for the optimization of a linear objective function under linear equality and inequality constraints, is embedded within this larger theoretical framework. Since George Dantzig's introduction of the simplex method, LP has become the workhorse of resource allocation, production planning, and logistics, offering precise solutions whenever problem parameters can be known exactly [34]. However, the crisp assumptions underlying classical LP restrict its applicability: real-world data often bear imprecision, vagueness, or inherent uncertainty that crisp coefficients and constraints cannot capture. This recognition has motivated extensions of decision theory capable of accommodating ambiguity, giving rise to fuzzy decision-making models that leverage Lotfi Zadeh's pioneering concept of fuzzy sets to represent partial truth and graded membership [13]. By permitting parameters to be expressed as membership functions rather than point values, fuzzy decision-making frameworks provide a structured means of handling ill-defined or incomplete information, thus broadening the practical scope of optimization under uncertainty [14, 15, 30].

Fuzzy linear programming (FLP) represents the natural confluence of fuzzy decision theory and the LP paradigm, extending classical LP by modeling coefficients, constraints,

and sometimes even the objective as fuzzy numbers [20]. In this way, FLP accommodates imprecision directly within the optimization process, translating vague problem descriptions into tractable mathematical programs. Early work in FLP explored symmetric formulations—where both objectives and constraints possess fuzzy membership functions—to ensure a balanced treatment of uncertainty throughout the model [22]. Complementing this, formulations with crisp objective functions but fuzzy constraints address scenarios in which goals are clearly articulated but the feasible region is uncertain, such as in resource allocation problems with fluctuating demand [23]. These foundational variants laid the groundwork for a rich taxonomy of FLP methods tailored to diverse application requirements and decision-maker preferences.

The significance of FLP is evident in its widespread application across multiple domains plagued by uncertainty. In energy management and water resources planning, FLP models optimize allocation under fuzzy supply and demand forecasts, yielding more resilient management policies than classical LP [24]. Manufacturing systems leverage FLP to accommodate imprecise demand projections and variable production capacities, thereby enhancing robustness in production planning. In financial portfolio optimization, fuzzy interest rate scenarios and market volatility are modeled through FLP to obtain asset allocations that respect investors' tolerance for risk and ambiguity [25]. Transportation and supply chain management further exploit FLP for routing and inventory decisions under uncertain delivery schedules and supplier reliability [26]. More recently, research has advanced parametric FLP approaches that systematically vary fuzzy coefficients to trace efficient frontiers, while hybrid techniques incorporating neural networks and genetic algorithms have been developed to mitigate computational complexity and improve solution robustness [27, 28].

Despite these advances, FLP faces several limitations that constrain its broader adoption. The incorporation of fuzzy sets into optimization increases computational burden, particularly for large-scale problems or those requiring high-resolution membership functions. Moreover, the interpretation of FLP solutions hinges on the shape and parameters of membership functions, demanding careful elicitation of decision-maker preferences and sensitivity analyses to ensure practical relevance. These challenges underscore the need for comparative evaluations of FLP methods, as different formulations and solution algo-

rithms may exhibit substantial variation in computational efficiency, solution quality, and ease of implementation.

In this dissertation, we undertake a comprehensive comparative study of FLP methodologies, encompassing symmetric and asymmetric formulations, parametric algorithms, and hybrid techniques. By synthesizing existing literature and conducting systematic numerical experiments, we evaluate each method's computational performance, solution robustness, and suitability for different classes of decision problems. Our analysis aims to elucidate the trade-offs inherent in various FLP approaches and to develop guidelines for selecting the most appropriate method given problem size, uncertainty structure, and decision-maker requirements. Through this work, we contribute to the ongoing refinement of fuzzy optimization techniques and provide practitioners with actionable insights for enhancing decision-making in complex, uncertain environments.

1.1 Organization of thesis

The structure of this dissertation is as follows: Chapter 2 contains basic concepts of the fuzzy sets, fuzzy terminologies and set theoretic operations. Classical Decision Theory, Decision Making in Fuzzy Environment, Fuzzy linear Programming and Symmetric FLP is discussed in Chapter 3. A solution methodology for Symmetric FLLP is also discussed in this chapter. In the next chapter, Chapter 4, Fuzzy LPP with crisp Objective Function is discussed. Further a numerical example is solved using a solution methodology studied for ZLPP with crisp Objective function and fuzzy constraints.

2 Basic concepts of Fuzzy Set

2.1 Crisp set

Let X be universal set. A crisp set [1] 'A' is defined as collection of well defined objects. A crisp set can be finite, countable or uncountable. In a crisp set, partial membership does not exist. It is defined by a function, known as characteristic function that determines which element of X are members of the set 'A' and which are not. The characteristic function $\chi_A(x)$ is defined as follows:

$$\chi_A(x) = \begin{cases} 1, & \text{when } x \in A \\ 0, & \text{when } x \notin A \end{cases}$$

Example 2.1 *Consider A = "set of tall persons with height greater than or equal to 140 cm". Then, persons with height 140 cm or more are tall and have membership value 1 and persons with height less than 140 cm have membership value 0.*

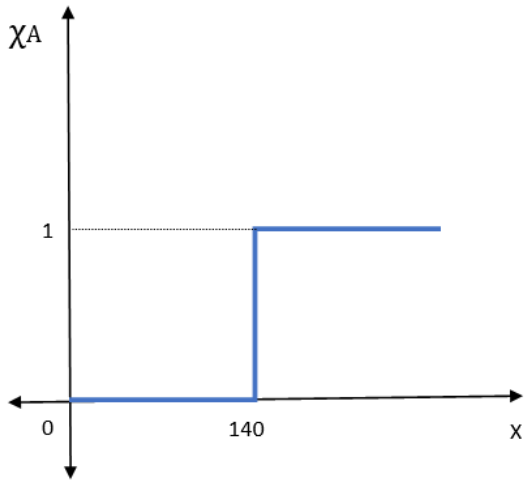

Figure 1: Crisp membership function

2.2 Fuzzy Set

Let x represent the universe of discourse, and let x represent any specific element in X. then a fuzzy set $\tilde{B}$ in the universal set X is defined as collection of ordered pairs[1] i.e.,

$$\tilde{B} = \{ (x, \mu_{\tilde{B}}(x)) \mid x \in X \; ; \; \mu_{\tilde{B}}(x) \in [0,1] \}$$

where $\mu_{\tilde{B}}(x)$ is membership function defined as $\mu_{\tilde{B}} : X \to [0,1]$. The first term in the ordered pair defined in the fuzzy set $\tilde{B}$ denotes the element of universal set and second term denotes the corresponding membership degree.

Example 2.2 *Let $X = \{p,q,r,s\}$ be the workers' reference set. Let $\tilde{B}$ denotes the fuzzy set of "labourious" workers. Then $\tilde{B}$ can be defined as :*

$$\tilde{B} = \{ (p,0.2),(q,0.6),(r,0.8),(s,1)\}.$$

Example 2.3 *Suppose $\tilde{B}$ be a fuzzy set representing "real numbers close to 10" with membership function*

$$\mu_{\tilde{B}}(x) = \frac{1}{1+(x-10)^4}, \quad x \in \mathbb{R}$$

shown in the following figure :

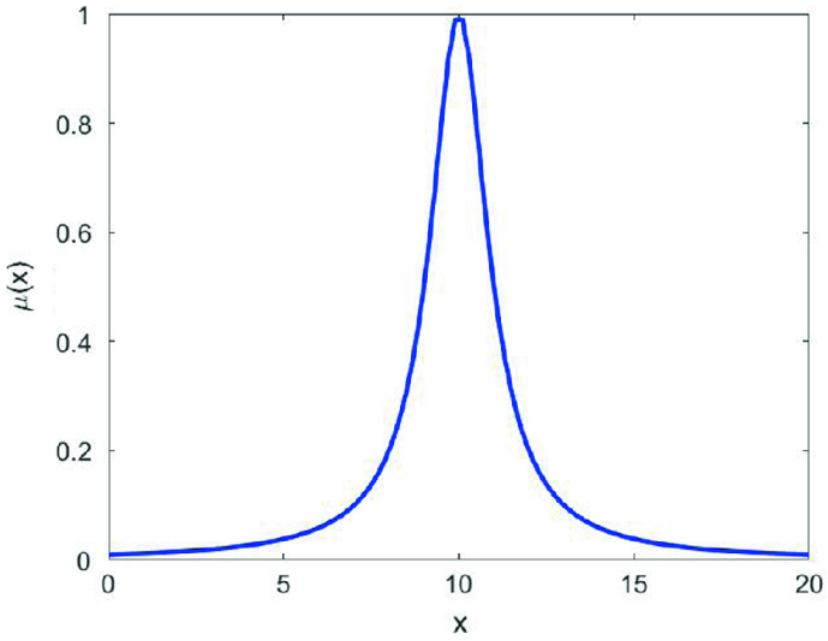

Figure 2: Real nos. nearly 10

2.3 Fuzzy Terminologies

2.3.1 Support

For any fuzzy set $\tilde{B}$, the set consists of all those $x \in X$ for which membership grade $\mu_{\tilde{B}}(x) > 0$ is called support of $\tilde{B}$.

In other words,

$$support(\tilde{B}) = \{x \in X | \mu_{\tilde{B}}(x) > 0\}$$

It can be graphically represented as:

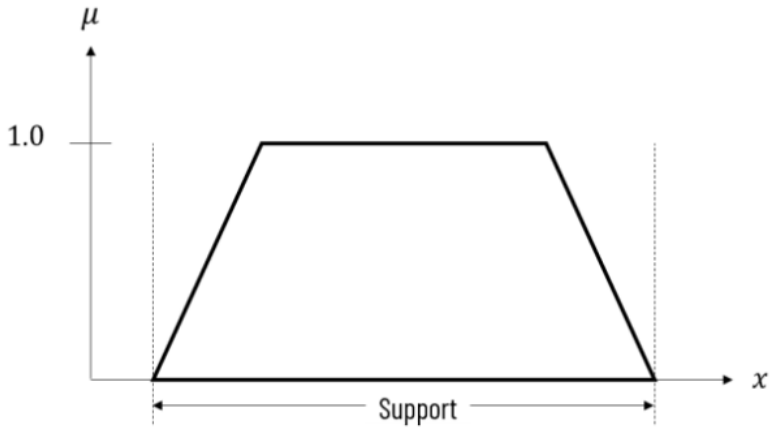

Figure 3: Support of fuzzy set

2.3.2 Core

Suppose $\tilde{B}$ be a fuzzy set. Then, the set consists of all those $x \in X$ for which membership grade $\mu_{\tilde{B}}(x) = 1$ is called core of $\tilde{B}$ i.e.,

$$core(\tilde{B}) = \{x \in X | \mu_{\tilde{B}}(x) = 1\}.$$

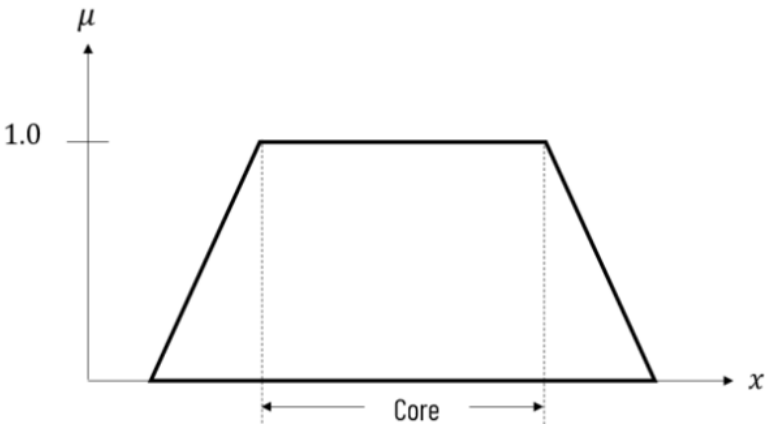

Figure 4: Core of fuzzy set

2.3.3 Height

For any fuzzy set $\tilde{B}$, the maximum membership value for the element consisting in the set $\tilde{B}$ is called height of fuzzy set $\tilde{B}$ i.e.,

$$height(\tilde{B}) = max\{\mu_{\tilde{B}}(x)\}.$$

It is not necessary that the height of a fuzzy set is always 1. It can be less than 1. For the height to be 1, the core of the fuzzy set must be non-empty.

2.3.4 Boundary

Let $\tilde{B}$ be a fuzzy set. Then, the set consists of all those elements for which membership grades $\mu_{\tilde{B}}(x)$ is greater than 0 and less than 1 is called boundary of fuzzy set $\tilde{B}$ i.e.,

$$boundary(\tilde{B}) = \{x \in X | 0 < \mu_{\tilde{B}}(x) < 1\}$$

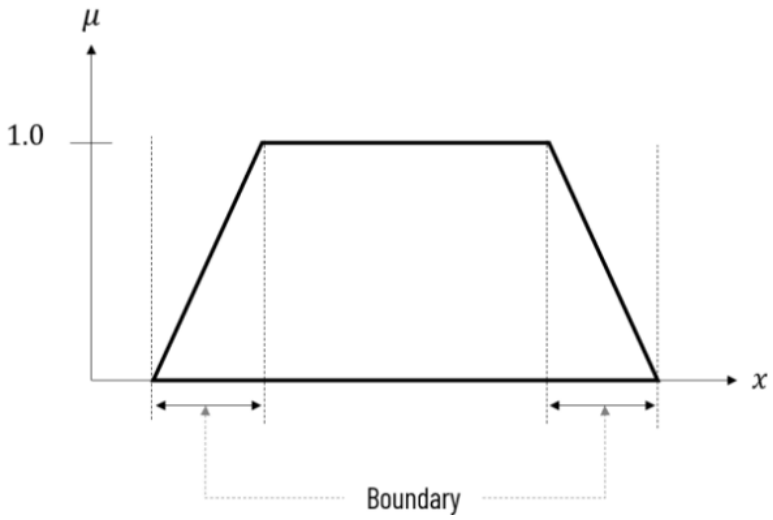

Figure 5: Boundary of fuzzy set

Example 2.4 *Let $X = \mathbb{N}$ be the universal set. Let a fuzzy set $\tilde{B}$ be given as*

$$\tilde{B} = \{(1,0.3),(2,0.4),(3,0.6),(4,0.2),(5,0.8),(6,0.7),(7,0.1)(8,0.5),(9,1),(10,0)\}$$

Then,

$$support(\tilde{B}) = \{1,2,3,4,5,6,7,8,9\}$$

$$core(\tilde{B}) = \{9\}$$

$$height(\tilde{B}) = 1$$

$$boundary(\tilde{B}) = \{1,2,3,4,5,6,7,8\}$$

2.3.5 Normal Fuzzy Set

Let $\tilde{B}$ be any fuzzy set. The fuzzy set $\tilde{B}$ is referred to as a normal fuzzy set if its height is 1. Graphically, it can be represented as

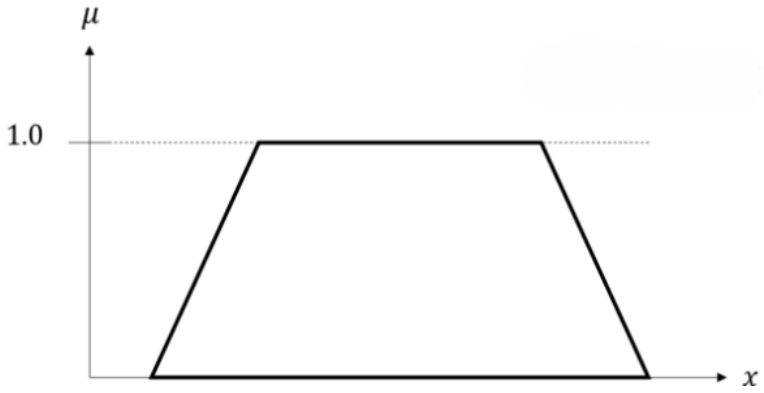

Figure 6: Normal fuzzy set

Example 2.5 *Consider a fuzzy set* $\tilde{B}$ *as*

$$\tilde{B} = \{(a,0.3),(b,0.6),(c,0.5),(d,1),(e,0.9)\}.$$

Clearly, $\tilde{B}$ *is normal fuzzy set, since* $height(\tilde{B}) = 1$.

2.3.6 α-cut and strong α-cut

Le $\tilde{B}$ be any fuzzy set and let $\alpha \in [0,1]$. Then, the set consisting the elements for which $\mu_{\tilde{B}}(x) \geqslant \alpha$ is called α-cut set, i.e.,

$$\tilde{B}_{\alpha} = \{x \in X | \mu_{\tilde{B}}(x) \geqslant \alpha\}.$$

For a fuzzy set $\tilde{B}$, if $\mu_{\tilde{B}}(x) > \alpha$, where $\alpha \in [0,1]$, the set is called strong α-cut i.e.,

$$\tilde{B}_{+\alpha} = \{x \in X | \mu_{\tilde{B}}(x) > \alpha\}.$$

Example 2.6 *Examine a fuzzy set.*

$$\tilde{B} = \{(1,0.3),(2,0.4),(3,1),(4,0.8),(5,0.9)\}$$

Then,

$$\tilde{B}_{0.3} = \{1,2,3,4,5\}$$

$$\tilde{B}_{0.4} = \{2,3,4,5\}$$

$$\tilde{B}_{+0.3} = \{2,3,4,5\}$$

$$\tilde{B}_{+0.9} = \{3\}$$

2.3.7 Convex Fuzzy set

Suppose X be universe of discourse and $\tilde{B}$ be any fuzzy set. Let $x_1,\ x_2 \in X$ and $\gamma \in [0,1]$, if $x_1,\ x_2$ satisfy the condition

$$\mu_{\tilde{B}}(\gamma x_1 + (1-\gamma)x_2) \geq min\{\mu_{\tilde{B}}(x_1), \mu_{\tilde{B}}(x_2)\},$$

then $\tilde{B}$ is called convex fuzzy set. the subsequent figure represents convex and non-convex fuzzy set

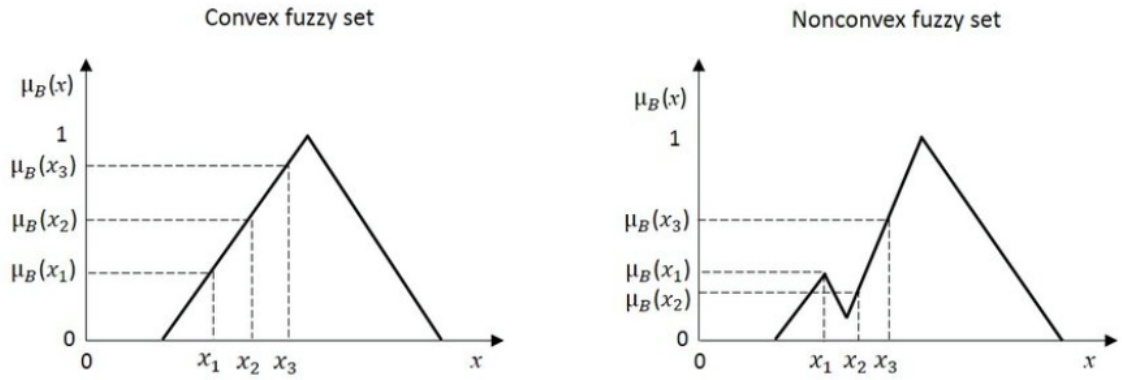

Figure 7: Convex and Non-convex fuzzy set

2.3.8 Cardinality of Fuzzy Set

Let $\tilde{B}$ be a finite fuzzy set. Then, cardinality of $\tilde{B}$ is defined as:

$$|\tilde{B}| = \sum_{x \in X} \mu_{\tilde{B}}(x)$$

For a fuzzy set $\tilde{B}$, the relative cardinality is defined as:

$$\|\tilde{B}\| = \frac{|\tilde{B}|}{|X|}$$

2.4 Standard Set Operations

The membership function has an important role in the operation of fuzzy sets. For any two fuzzy sets $\tilde{B}$ and $\tilde{C}$, membership function for union of $\tilde{B}$ and $\tilde{C}$ involves maximum of membership functions of both sets while intersection involves minimum and compliment involves subtraction of the membership function from 1.

Let X be the universal set and $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) \mid x \in X\}$ and $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) \mid x \in X\}$ be two fuzzy sets defined over X. Then, set theoretic operations on fuzzy sets are defined as follows:

Union

Union of fuzzy sets $\tilde{B}$ and $\tilde{C}$ is defined using maximum of membership grades i.e.,

$$\tilde{B} \cup \tilde{C} = \{(x, \mu_{\tilde{B} \cup \tilde{C}}(x)) | \ x \in X\}$$

where

$$\mu_{\tilde{B} \cup \tilde{C}}(x) = max\{\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\}, \qquad \forall x \in X$$

Intersection

Intersection of fuzzy sets $\tilde{B}$ and $\tilde{C}$ is defined using minimum of membership grades i.e.,

$$\tilde{B} \cap \tilde{C} = \{(x, \mu_{\tilde{B} \cup \tilde{C}}(x)) | \ x \in X\}$$

where

$$\mu_{\tilde{B} \cap \tilde{C}}(x) = min\{\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\}, \qquad \forall x \in X$$

Compliment

Compliment of fuzzy set $\tilde{B}$ is defined as:

$$\tilde{B}' = \{(x, \mu_{\tilde{B}'}(x)) | \ x \in X\}$$

where

$$\mu_{\tilde{B}'}(x) = 1 - \mu_{\tilde{B}}(x), \qquad \forall x \in X$$

Equality

Two fuzzy sets $\tilde{B}$ and $\tilde{C}$ are said to be equal if and only if their corresponding membership grades are equal i.e.,

$$\mu_{\tilde{B}}(x) = \mu_{\tilde{C}}(x) \qquad \forall x \in X.$$

Example 2.7 *Consider two fuzzy sets*

$$\tilde{B} = \{\,(a,0.4),(b,0.6),(c,0.8),(d,1)\}$$

and

$$\tilde{C} = \{\,(a,0.1),(b,0.5),(c,0.3),(d,0)\}.$$

Then,

$$\tilde{B} \cup \tilde{C} = \{\,(a,0.4),(b,0.6),(c,0.8),(d,1)\},$$

$$\tilde{B} \cap \tilde{C} = \{\,(a,0.1),(b,0.5),(c,0.3),(d,0)\},$$

$$\tilde{B}' = \{\,(a,0.6),(b,0.4),(c,0.2),(d,0)\}.$$

3 Decision Making in Fuzzy Enviroments

3.1 Classical Decision Theory

Definition 3.1 *A decision is defined by*

a) *a collection of decision alternative (decision space),*

b) *a collection of natural states (the state space),*

c) *a relation assigned to each pair of decisions and state a result,*

d) *Ultimately, the utility function orders the outcomes based on their desirability.*

There are two environments for decision-making:

- Given a dominant condition of nature, the decision maker chooses the option with the maximum utility while making a decision under certainty since they know which state to anticipate.
- The decision maker only knows the probability function of the state while making decision under risk; they do not know exactlly which state will occur.

Example 3.1 *Assume for the moment that the directors' board desires to determine the ideal profit share, and that maximizing the payout is their goal (utility function). The profit share, which ranges from 0 to 6 percent, is the constraint that defines the decision space. Therefore, a dividend of* 6 *percent will be ideal. The following figure shows the relationships by assigning a linear utility function.*

3.1.1 Fuzzy Decision

Bellman and Zadeh [2] examined the traditional model of decision-making in 1970 and proposed a model for making decisions in a fuzzy setting. They reasoned the following in a scenario involving decision-making under certainty where the fuzzy goal and the constraints are ambiguous. Both the constraints and the fuzzy goal are defined by their membership functions.

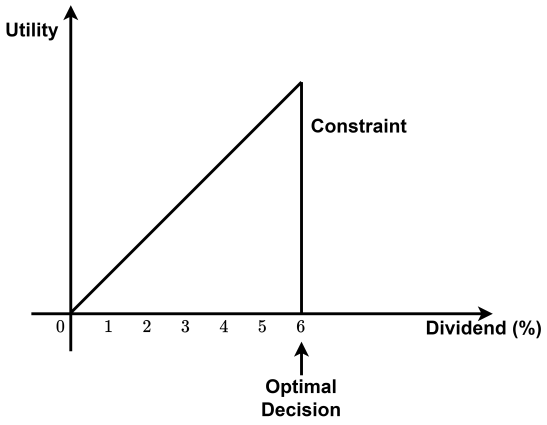

Figure 8: A Classical Decision Under Certainty

Definition 3.2 *In a set of options Y, let's examine a fuzzy objective function (or fuzzy aim) $\tilde{A}$ and a fuzzy constraint $\tilde{B}$. Next, $\tilde{C}$, termed* ***fuzzy decision****, is a fuzzy set that is the consequence of $\tilde{A}$ and $\tilde{B}$ intersecting. symbolically, $\tilde{C}$ equals $\tilde{A} \cap \tilde{B}$ and $\mu_{\tilde{C}} = min\{\mu_{\tilde{A}}, \mu_{\tilde{B}}\}$ correspondingly. More broadly, let's say we have 'm' constraints $\tilde{B_1}$, $\tilde{B_2}$,...,$\tilde{B_m}$, and 'n' fuzzy objective functions $\tilde{A_1}$, $\tilde{A_2}$,...,$\tilde{A_n}$. The intersection of the provided fuzzy objectives and constraints is the fuzzy decision defined as:*

$$\tilde{C} = \tilde{A_1} \cap \tilde{A_2} ... \cap \tilde{A_n} \cap \tilde{B_1} \cap \tilde{B_2} ... \tilde{B_m}$$

and

$$\mu_{\tilde{C}} = min\{\mu_{\tilde{A_i}}, \mu_{\tilde{B_j}}\}.$$

Example 3.2 *Let us consider an example where the objective function $\tilde{O}$ is given as "y larger than 10 " defined by the membership function*

$$\mu_{\tilde{O}}(y) = \begin{cases} 0 & ,y \leq 10 \\ (1+(y-10)^{-2})^{-1} & ,y > 10 \end{cases}$$

and the constraint $\tilde{C}$ is given as "y in the vicinity of 11" defined by the membership function

$$\mu_{\tilde{C}}(y) = (1+(y-11)^4)^{-1}$$

The membership function $\mu_{\tilde{D}}(y)$ of the decision is then

$$\mu_{\tilde{D}}(y) = \begin{cases} min\{(1+(y-10)^{-2})^{-1}, (1+(y-11)^4)^{-1}\} & ,y > 10 \\ 0 & ,y \leq 10 \end{cases}$$

$$\Longrightarrow \quad \mu_{\tilde{D}}(y) = \begin{cases} 0 & ,y \leq 10 \\ (1+(y-10)^{-2})^{-1} & ,10 < y < 11.7549 \\ (1+(y-11)^4)^{-1} & ,y > 11.7549 \end{cases}$$

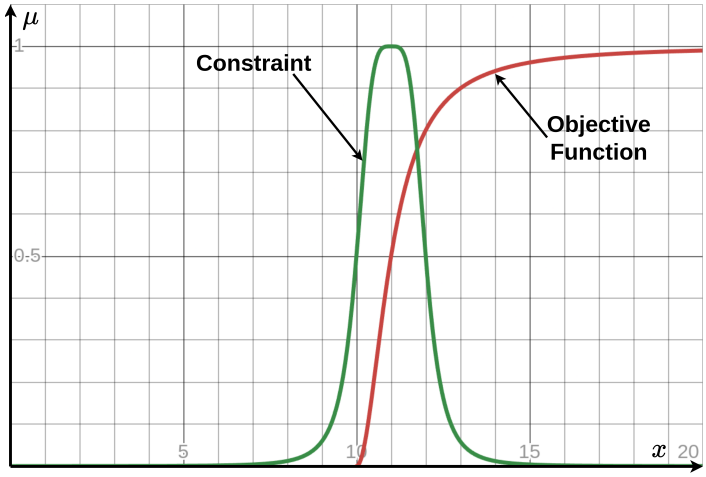

Figure 9: Fuzzy Decision for **Example 3.2**

3.1.2 Fuzzy Linear Programming

Models of linear programming will be regarded as a unique type of decision model.

1) The constraints define the decision space.

2) The goal (utility function) is defined by the objective function, and

3) The kind of decision is one that is made with certainty.

The classical model of LPP can be stated as

$$\begin{aligned} \max \quad & f(y) = C^T y, \\ \text{such that} \quad & Ay \leq b, \\ & y \geq 0 \end{aligned} \tag{1}$$

and C, $y \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. In model (1), all coefficients A, b and C are crisp, '$\leq$' is meant in a crisp sense and 'maximize' is a strict imperative.There are few modifications that can be done in (1) if we consider that the LP choice must be made in a fuzzy environment.

3.1.3 Symmetric Fuzzy LP

[2] Now, let's move on to the first fundamental model for fuzzy linear programming. Assuming that the decision-maker can select an aspiration level z for the value of the objective function they wish to attain and that each constraint is represented as a fuzzy set, the Symmetric FLP' in model (1) can be defined as follows: Find y such that

$$\begin{aligned} & C^T y \tilde{\geq} z \\ \text{subject to} \quad & Ay \tilde{\leq} b \\ & y \geq 0 \end{aligned} \tag{2}$$

Model (2) is fully symmetric with respect to objective function and constraints, we want to make that even more obvious by substituting

$$\begin{pmatrix} -C \\ A \end{pmatrix} = B \text{ and } \begin{pmatrix} -z \\ b \end{pmatrix} = d$$

Then model (2) become

$$\begin{aligned} By &\tilde{\leq} d \\ y &\geq 0 \end{aligned} \tag{3}$$

Each of (m+1) rows of model (3) shall now be represented by a fuzzy set, defined by the membership function $\mu_i(y)$, $i = 1, 2, ..., m+1$.

The membership function of the "decision" for model (3) is

$$\mu_{\tilde{D}}(y) = \min \mu_i(y) \; \forall i = 1, 2, ..., m+1. \tag{4}$$

$\mu_i(y)$ can be interpreted as the degree to which 'y' fulfills (satisfies) the fuzzy inequality $B_i y \leq d_i$ (where B_i denotes the i^{th} row of B).

Given the assumption that the decision maker prefers a crisp"optimal" answer , we might propose the "maximizing solution" to eq.(4), which is the solution to the possibly nonlinear programming problem.

$$\max \; \min \mu_i(y) = \max \; \; \mu_{\tilde{D}}(y) \quad \forall y \geq 0. \tag{5}$$

The membership function $\mu_i(y)$ must now be specified. If the constraint (including the goal function) is substantially violated, $\mu_i(y)$ should be 0; if it is very well satisfied, it should be 1. Thus, $\mu_i(y)$ can be described as follows:

$$\mu_i(y) = \begin{cases} 1 & ,B_i y \leq d_i \\ \in [0,1] & ,d_i < B_i y \leq d_i + p_i \\ 0 & ,B_i y > d_i + p_i \end{cases} \tag{6}$$

Using the most basic kind of membership function, we assume that they increase linearly across the tolerance interval.p_i, i.e.,

$$\mu_i(y) = \begin{cases} 1 & ,B_i y \leq d_i \\ 1 - \frac{B_i y - d_i}{p_i} & ,d_i < B_i y \leq d_i + p_i \\ 0 & ,B_i y > d_i + p_i \end{cases} \tag{7}$$

The constantsp_i are subjectively selected and represent admissible violations of the objective function and the constraint.

Substituting eq.(7) into problem (5) yields, after some rearrangements and with some additional assumption,

$$\max \ \min \left(1 - \frac{B_i y - d_i}{p_i} \right) \ \forall y \geq 0 \text{ and } \forall i \tag{8}$$

Introducing, one new variable λ which corresponds essentially to eq.(4) we arrive at the following problem:

$$\begin{array}{ll} \max & \lambda \\ \text{such that} & \lambda p_i + B_i y \leq d_i + P_i, \quad i = 1,2,\ldots,m+1, \\ & \forall y \geq 0. \end{array} \tag{9}$$

If the optimal solution to problem (9) is a vector (λ, y_0) then y_0 is the maximizing solution of (5) of model (2) assuming membership function as specified in (7).

Example 3.3 *[2] A business needed to make decisions about the composition and size of its fleet of trucks. Four trucks of varying sizes (y_1 through y_4) were taken into consideration. Cost reduction was the goal and meeting the demands of all clients—whose demand varies significantly with the seasons—was the limitation. Due to quantity limits, some quantities*

had to be shifted, and a routing constraint required that a minimum number of consumers be contacted each day. At least six of the smallest vehicles had to be part of the fleet for various reasons. The management accepted the following recommended linear programming approach and desired to employ quantitative analysis:

$$\begin{aligned} \min \quad & 41400y_1 + 44300y_2 + 48100y_3 + 49100y_4 \\ \textit{subject to constraints} & \\ & 0.84y_1 + 1.44y_2 + 2.16y_3 + 2.4y_4 \geq 170 \\ & 16y_1 + 16y_2 + 16y_3 + 16y_4 \geq 1300 \\ & y_1 \geq 6 \\ & y_2, y_3, y_4 \geq 0 \end{aligned}$$

The solution was $y_1 = 6,\ y_2 = 16.29,\ y_3 = 0,\ y_4 = 58.96$, *and MinCost = 3864975. Upon presenting the results to management, it was discovered that while the findings seemed acceptable, they would prefer to have some "leeway" in the limits.*

The cost of $42,00,000$ *is shown in the budget and cannot be exceeded. The model (3) was chosen to correctly represent management's perspectives of the issue since it made management feel compelled to provide exact limitations (despite the fact that it would have preferred to provide some intervals). The parameters listed below were estimated:*

Lower bounds of the tolerance interval:

$d_1 = 3700000,\ d_2 = 70,\ d_3 = 300,\ d_4 = 6.$

Spreads of the tolerance intervals:

$p_1 = 500000,\ p_2 = 10,\ p_3 = 100,\ p_4 = 6.$

Consider the membership function $\mu_i(y)$ $(for\ i = 1)$ *for the objective function as follows:*

$$\mu_i(y) = \begin{cases} 1 & ,B_i y \leq d_i \\ 1 - \frac{B_i y - d_i}{p_i} & ,d_i < B_i y \leq d_i + p_i \\ 0 & ,B_i y > d_i + p_i \end{cases}$$

Take

$$\lambda \leq \mu_i(y) = 1 - \frac{B_i y - d_i}{p_i},$$

After some calculation, we get:

$$\frac{B_i y}{p_i} + \lambda \leq \frac{d_i}{p_i} + 1 \tag{10}$$

Using eq.(10) for the objective function, we get::

$$\frac{1}{500000}(41400y_1+44300y_2+48100y_3+49100y_4)+\lambda \leq \frac{3700000}{500000}+1$$

$$\implies 0.083y_1+0.039y_2+0.096y_3+0.098y_4+\lambda \leq 8.4$$

Now consider the following membership function $\mu_i(x)$, *(for* $i=2,3,4$*) for the constraints:*

$$\mu_i(y)=\begin{cases}1 & ,B_iy \geq d_i \\ \frac{B_iy-d_i}{p_i} & ,d_i-p_i \leq B_iy < d_i \\ 0 & ,B_iy < d_i-p_i\end{cases}$$

Take

$$\lambda \leq \mu_i(y)=\frac{B_iy-d_i}{p_i}$$

After some calculations, we get

$$\frac{B_iy}{p_i}-\lambda \geq \frac{d_i}{p_i} \tag{11}$$

Using Eq.(11) for the first constraint, we get:

$$\frac{1}{10}(0.84y_1+1.44y_2+2.16y_3+2.4y_4)-\lambda \geq \frac{170}{10}$$

$$\implies \quad 0.084y_1+0.144y_2+0.216y_3+0.24y_4-\lambda \leq 17$$

Similarly applying Eq.(11) on the rest of the constraints, we get,

$$0.16y_1+0.16y_2+0.16y_3+0.16y_4-\lambda \geq 13$$

$$0.167y_1-\lambda \geq 1$$

So we have final model to solve as:

$$\begin{aligned}\max \quad & \lambda \\ \textit{subject to} \quad & 0.083y_1+0.039y_2+0.096y_3+0.098y_4+\lambda \leq 8.4 \\ & 0.084y_1+0.144y_2+0.216y_3+0.24y_4-\lambda \leq 17 \\ & 0.16y_1+0.16y_2+0.16y_3+0.16y_4-\lambda \geq 13 \\ & 0.167y_1-\lambda \geq 1 \\ & \lambda, y_1, y_2, y_3, y_4 \geq 0\end{aligned} \tag{12}$$

After solving LPP(12), we get:

$y_1=17.414,\ y_2=0,\ y_3=0,\ y_4=66.54$ *and* $z=39,88,250.$

As can be seen from the solution, "leeway" has been provided with respect to all constraints and at additional cost of 3.2 percent .

4 Fuzzy LP With Crisp Objective Function

- In model (2) we see that both the objective function and the constraints are fuzzy so we apply equation (3)-(5) to solve it.
- When the constraints are all or some of them are fuzzy and has a crisp objective function—that is, one that must be maximized or minimized—no longer has symmetry. The approach (3)-(5) is therefore unsuccessful.
- In order to solve such kind of problem, we will create the concept of maximizing set, which we will utilize to define the problem of determining an extremum of a crisp function over a fuzzy domain.

Definition 4.1 *Let f be a real-valued function in Y. Let f be bounded from below by* $\inf(f)$ *and from above by* $\sup(f)$ *then the fuzzy set* $\tilde{M} = \{(y, \mu_{\tilde{M}}(y))\}$ *where* $y \in Y$ *with*

$$\mu_{\tilde{M}}(y) = \frac{f(y) - \inf f}{\sup f - \inf f}$$

is called the 'maximizing set'.

Two approaches are conceivable (for fuzzy LP with crisp objective function):

1) The process of determining the fuzzy set "decision."

2) By aggregating the objective function following the necessary transformations with the restrictions, a crisp "maximizing decision" is determined.

4.1 The Determination of Fuzzy Set "Decision"

With the degree of membership equal to the corresponding α-level of the solution space, the fuzzy set "decision" is defined as the optimal values of the objective functions. Orlovski suggests calculating the goal function's associated optimal values for each α-level set in the solution space.

Definition 4.2 *[2] Let* $R_\alpha = \{y | y \in Y, \mu_R(y) \geq \alpha\}$ *be α-level sets of the solution space and* $N(\alpha) = \{y | y \in R_\alpha, f(y) = \sup_{y' \in R_\alpha} f(y')\}$ *the set of optimal solution for α-level set. The fuzzy*

set "decision" is then defined by the membership function

$$\mu_{opt}(y) = \begin{cases} \sup \alpha & ,if \ \ y \in \bigcup\limits_{\alpha \geq 0} N(\alpha) \\ 0 & ,otherwise \end{cases}$$

The fuzzy set "optimal values of the objective function" has the membership function

$$\mu_f(r) = \begin{cases} \sup\limits_{y \in f^{-r}} \mu_{\text{opt}}(y) & ,\text{if } r \in R_1 \text{ and } f^{-r} \neq \emptyset \\ 0 & ,\text{otherwise} \end{cases}$$

Here $f(y)$ is objective function with functional value r.

For the case of linear programming, the determination of the r's and $\mu_{opt}(y)$ can be obtained by parametric programming .

Consider the LP$-$ Model

$$\begin{aligned} \max \quad & Cy \\ \text{such that} \quad & \sum_{j=1}^{n} a_{ij} y_j \tilde{\leq} b_i \quad i = 1,2,\ldots,m, \end{aligned}$$

then parametric programming problem is given by

$$\begin{aligned} \max \quad & Cy \\ \text{such that} \quad & \sum_{j=1}^{n} a_{ij} y_j \leq b_i + \theta p_i \end{aligned}$$

where b_i is the aspiration level, p_i is the greatest value of violation, and θ is the parameter. The set of solutions maximizing the objective function as analytically dependent on the parameter θ is obtained by solving the problem using the parametric programming technique. That is, for every θ, we obtain a solution that simultaneously achieves the goal in the possibly highest degree while satisfying "jointly" the constraint with degree $1-\theta$.

Example 4.1 *Consider the problem [3]*

$$\begin{aligned} \max \quad & Z = 3y_1 + 4y_2 + 4y_3 \\ s.t. \quad & 6y_1 + 3y_2 + 4y_3 \tilde{\leq} 1200 \\ & 5y_1 + 4y_2 + 5y_3 \tilde{\leq} 1550 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \tag{13}$$

Let $b_0 = 1700$, $p_0 = 150$, $p_1 = 100$, $p_2 = 200$.

We convert the problem (13) into the following parametric programming problem (14)

$$\begin{aligned} \max \quad & Z = 3y_1 + 4y_2 + 4y_3 \\ s.t. \quad & 6y_1 + 3y_2 + 4y_3 \leq 1200 + 100\theta \\ & 5y_1 + 4y_2 + 5y_3 \leq 1550 + 200\theta \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \tag{14}$$

- *when* $\theta = 0$ *the problem reduces to a standard LP Problem.*
- *when* θ *changes the feasible region changes.*
- *different values of* θ *lead to different optimal solution.*

We obtain the following result

$$\theta_y = \begin{cases} (0, \frac{775}{2} + 50\theta, 0) & ,0 \leq \theta \leq \frac{3}{8} \\ (0, \frac{1575}{4} + \frac{100}{3}\theta, 0) & ,\frac{3}{8} \leq \theta \leq 1 \end{cases}$$

$$z(\theta_y) = \begin{cases} 1550 + 200\theta & ,0 \leq \theta \leq \frac{3}{8} \\ 1575 + \frac{400}{3\theta} & ,\frac{3}{8} \leq \theta \leq 1 \end{cases}$$

$$\mu_0(\theta_y) = \begin{cases} \frac{4}{3}\theta & ,0 \leq \theta \leq \frac{3}{8} \\ \frac{1}{6} + \frac{8}{9\theta} & ,\frac{3}{8} \leq \theta \leq \frac{15}{16} \\ 1 & ,\theta \geq \frac{15}{16} \end{cases}$$

After solving, we get that $\theta = \frac{15}{34}$ *determines the maximizing solution (0, 408.44, 0). Figure 10 represents the graph of function* $\mu_c(\theta_y)$, $\mu_o(\theta_y)$ *and* $\mu_D(\theta_y)$. *The maximal value of the function* $\mu_D(\theta_y)$ *equals* $\frac{29}{51}$ *and is attained for* $\theta = \frac{15}{34}$ *which determines the maximizing solution (0, 408.44, 0).*

- This approach helps decision makers understand how changing in resources (θ) affects the optimal solution.
- Used in sensitivity analysis for real word applications like supply chain management, production planning and finance.

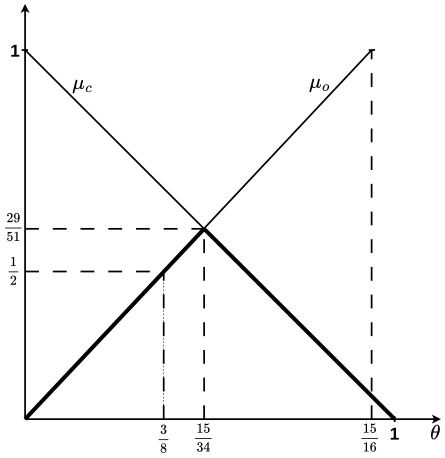

Figure 10: Fuzzy Decision for **Example 4.1**

4.1.1 The Determination of a Crisp Maximizing Decision

Definition 4.3 *[2] Let $f : Y \to R^1$ be the objective function, $\tilde{R}$ = fuzzy feasible region, $S(\tilde{R})$ = support of $\tilde{R}$, and $R_1 = \alpha-$ level cut of $\tilde{R}$ for $\alpha = 1$. The membership function of the goal (objective function) given solution space $\tilde{R}$ is then defined as*

$$\mu_{\tilde{G}}(y) = \begin{cases} 0 & , \ f(y) \leq \sup\limits_{R_1} f \\ \dfrac{f(y) - \sup\limits_{R_1} f}{\sup\limits_{S(\tilde{R})} f - \sup\limits_{R_1} f} & , \ \sup\limits_{R_1} f \leq f(y) \leq \sup\limits_{S(\tilde{R})} \\ 1 & , \ \sup\limits_{R_1} f \leq f(y) \end{cases}$$

The corresponding membership function in functional space is then

$$\mu_{\tilde{G}}(r) = \begin{cases} \sup\limits_{y \in f^{-1}(r)} \mu_{\tilde{G}}(y) & , \ y \in R, f^{-1}(r) \neq \emptyset \\ 0 & , \ otherwise \end{cases}$$

Example 4.2 *Consider the LP model*

$$\begin{aligned} \max \quad & z = 2y_1 + y_2 \\ s.t. \quad & y_1 \tilde{\leq} 3 \\ & y_1 + y_2 \tilde{\leq} 4 \\ & 5y_1 + y_2 \tilde{\leq} 3 \\ & y_1, y_2 \geq 0 \end{aligned} \tag{15}$$

The tolerance intervals of the constraints are $p_1 = 6,\ p_2 = 4,\ p_3 = 2.$

The parametric linear program for determining the relationships between $f(y) = r$ *and degree of membership is then defined as*

$$\begin{aligned} \max \quad & z = 2y_1 + y_2 \\ s.t. \quad & y_1 \leq 9 - 6\alpha \\ & y_1 + y_2 \leq 8 - 6\alpha \\ & 5y_1 + y_2 \leq 5 - 2\alpha \\ & y_1, y_2 \geq 0 \end{aligned} \tag{16}$$

For this model let R_1 *be the region defined by*

$$\begin{aligned} y_1 &\leq 9 \\ y_1 + y_2 &\leq 8 \\ 5y_1 + y_2 &\leq 5 \end{aligned} \tag{17}$$

The supremum of f over this region is

$$\sup_{R_1} 2y_1 + y_2 = 7$$

Using the min-max approach, the resulting solution is $y_1^0 = 5.84,\ y_2^0 = .05,\ r_0 = 11.73$*, and the attained degree of membership is* $\mu_{\bar{R}}(y_0) = 0.53.$

The crisp maximizing decision is the specific values of y_1 *and* y_2 *that yield the maximum while satisfying all the constrain*

5 Conclusion and Future work

In this dissertation, we have studied basics of fuzzy set theory and its applications, decision making in fuzzy environment, classical decision theory, fuzzy decision and fuzzy linear programming. We have studied fuzzy linear programming as a unique form of decision model. We have researched multiple approaches for various fuzzy linear programming in order to solve them and get optimal solution. For instance, we studied the model where the goal (utility function) and the constraints are both fuzzy, called symmetric FLPP. On the other hand we also explored the FLPP consisting crisp objective function and entirely or partially fuzzy constraints. To tackle the problem, we employ parametric programming or the simplex method. Our approach demonstrated a robust framework for handling imprecision and uncertainty in optimization problems. These methods assist decision makers in comprehending how resource changes impact the best possible outcome. This approach gives decision makers a more comprehensive and flexible toolkit and opens up new avenues for research into intricate optimization techniques in fuzzy environments. We intend to tackle various FLPP types as a future project.

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